COMPACTA WITH SHAPES OF FINITE COMPLEXES: A DIRECT APPROACH TO THE EDWARDS-GEOGHEGAN-WALL OBSTRUCTION

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Abstract. An important "stability" theorem in shape theory, due to D.A. Edwards and R. Geoghegan, characterizes those compacta having the same shape as a finite CW complex. In this note we present straightforward and self-contained proof of that theorem.

1. Introduction

Before Ross Geoghegan turned his attention to the main topic of these proceedings, *Topological Methods in Group Theory*, he was a leader in the area of shape theory. In fact, much of his pioneering work in geometric group theory has involved taking key ideas from shape theory and recasting them in the service of groups. Some of his early thoughts on that point of view are captured nicely in [Ge2]. Among the interesting ideas found in that 1986 paper is an early recognition that a group boundary is well-defined up to shape—an idea later formalized by Bestvina in [Be].

In this paper we return to the subject of Geoghegan's early work. For those whose interests lie primarily in group theory, the work presented here contains a concise and fairly gentle introduction to the ideas of shape theory, via a careful study of one of its foundational questions.

In the 1970's D.A. Edwards and R. Geoghegan solved two open problems in shape theory—both related to the issue of "stability". Roughly speaking, these problems ask when a "bad" space has the same shape as a "good" space. For simplicity, we focus on the following versions of those problems:

Problem A. *Give necessary and sufficient conditions for a connected finite-dimensional compactum Z to have the pointed shape of a CW complex.*

Problem B. *Give necessary and sufficient conditions for a connected finite-dimensional compactum Z to have the pointed shape of a finite CW complex.*

Solutions to these problems can be found in the sequence of papers: [EG1], [EG2], [EG3]. A pair of particularly nice versions of those solutions are as follows:

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Solution A. *Z has the pointed shape of a CW complex if and only if each of its homotopy pro-groups is stable.*

Solution B. *Z has the pointed shape of a finite CW complex if and only if each of its homotopy pro-groups is stable and an intrinsically defined Wall obstruction $\omega(Z,z) \in \widetilde{K}_0(\mathbb{Z}[\check{\pi}_1(Z,z)])$ vanishes.*

Solution B was obtained by combining Solution A with C.T.C. Wall's famous work on finite homotopy types [Wa]. So, in order to understand Edwards and Geoghegan's solution to Problem B, it is necessary to understand two things: Solution A; and Wall's work on the finiteness obstruction. Since both tasks are substantial—and since Problem B can arise quite naturally without regards to Problem A—we became interested in finding a simpler and more direct solution to Problem B. This note contains such a solution. This paper may be viewed as a sequel to [Ge1], where Geoghegan presented a new and more elementary solution to Problem A. In the same spirit, we feel that our work offers a simplified view of Problem B.

The strategy we use in attacking Problem B is straightforward and very natural. Given a connected n-dimensional pointed compactum Z, begin with an inverse system $K_0 \xleftarrow{f_1} K_1 \xleftarrow{f_2} K_2 \xleftarrow{f_3} \cdots$ of finite n-dimensional (pointed) complexes with (pointed) cellular bonding maps that represents Z. Under the assumption that pro-π_k is stable for all k, we borrow a technique from [Fe] allowing us to attach cells to the K_i's so that the bonding maps induce π_k-isomorphisms for increasingly large k. Our goal then is to reach a *finite* stage where the bonding maps induce π_k-isomorphisms for all k, and are therefore homotopy equivalences. This would imply that Z has the shape of any of those homotopy equivalent finite complexes. As expected, we confront an obstruction lying in the reduced projective class group of pro-π_1. Instead of invoking theorems from [Wa], we uncover this obstruction in the natural context of the problem at hand; in fact, the main result of [Wa] can then be obtained as a corollary. Another advantage to the approach taken here is that all CW complexes used in this paper are finite. This makes both the algebra and the shape theory more elementary.

2. Background

In this section we provide some background information on inverse systems, inverse sequences, and shape theory. In addition, we will review the definition of a reduced projective class group. A more complete treatment of inverse systems and sequences can be found in [Ge3]; an expanded version of this introduction can be found in [Gu]

2.1. **Inverse systems.** We provide a brief discussion of general inverse systems and pro-categories, which provide the broad framework for more concrete constructions that will follow. A thorough treatment of this topic can be found in [Ge3, Ch.11].

An *inverse system* $\left\{X_\alpha, f_\alpha^\beta; \mathcal{A}\right\}_{\alpha\in\mathcal{A}}$ consists of a collection of objects X_α from a category $\mathcal{C}$ indexed by a *directed set* $\mathcal{A}$, along with morphisms $f_\alpha^\beta : X_\beta \to X_\alpha$ for every pair $\alpha, \beta \in \mathcal{A}$ with $\alpha \le \beta$, satisfying the property that $f_\alpha^\gamma = f_\beta^\gamma \circ f_\alpha^\beta$ whenever

$\alpha \leq \beta \leq \gamma$. By fixing $\mathcal{C}$, but allowing the directed set to vary, and formulating an appropriate definition of morphisms, one obtains a category pro-$\mathcal{C}$ whose objects are all such inverse systems. When $\mathcal{A}' \subseteq \mathcal{A}$ is a directed set there is an obvious subsystem $\left\{X_\alpha, f_\alpha^\beta; \mathcal{A}'\right\}_{\alpha\in\mathcal{A}'}$ and an inclusion morphism. When $\mathcal{A}'$ is *cofinal* in $\mathcal{A}$ (for every $\alpha \in \mathcal{A}$ there exists $\alpha' \in \mathcal{A}'$ such that $\alpha \leq \alpha'$), the inclusion morphism is an isomorphism in pro-$\mathcal{C}$. A key theme in this subject is that, when $\mathcal{A}'$ is cofinal, the corresponding subsystem contains all relevant information.

When $\mathcal{C}$ is a category of sets and functions, we may define the *inverse limit* of $\left\{X_\alpha, f_\alpha^\beta; \mathcal{A}\right\}_{\alpha\in\mathcal{A}}$ by

$$\varprojlim \left\{X_\alpha, f_\alpha^\beta; \mathcal{A}\right\} = \left\{ (x_\alpha) \in \prod_{\alpha\in\mathcal{A}} X_\alpha \,\middle|\, f_\alpha^\beta\left(x_\beta\right) = x_\alpha \text{ for all } \alpha \leq \beta \right\}$$

along with projections $p_\alpha : \varprojlim \left\{X_\alpha, f_\alpha^\beta; \mathcal{A}\right\} \to X_\alpha$. When $\mathcal{C}$ is made up of topological spaces and maps, the inverse limits are topological spaces and the projections are continuous. Similarly, additional structure is passed along to inverse limits when $\mathcal{C}$ consists of groups, rings, or modules and corresponding homomorphisms. An important example of the "key theme" noted in the previous paragraph is that, when $\mathcal{A}'$ is is cofinal in $\mathcal{A}$, the canonical inclusion $\varprojlim \left\{X_\alpha, f_\alpha^\beta; \mathcal{A}'\right\} \to \varprojlim \left\{X_\alpha, f_\alpha^\beta; \mathcal{A}'\right\}$ is a bijection of sets [resp., homeomorphism of spaces, isomorphism of groups, etc.].

An *inverse sequence* (or *tower*) is an inverse system for which $\mathcal{A} = \mathbb{N}$, the natural numbers. Since all inverse systems used in this paper contain cofinal inverse sequences, we are able to work almost entirely with towers. General inverse systems play a useful, but mostly invisible, background role.

2.2. **Inverse sequences (aka towers).** The fundamental notions that make up a category pro-$\mathcal{C}$ are simpler and more intuitive when restricted the the subcategory of towers in $\mathcal{C}$. For our purposes, an understanding of towers will suffice; so that is where we focus our attention.

Let

$$C_0 \xleftarrow{\lambda_1} C_1 \xleftarrow{\lambda_2} C_2 \xleftarrow{\lambda_3} \cdots$$

be an inverse sequence in pro-$\mathcal{C}$. A *subsequence* of $\{C_i, \lambda_i\}$ is an inverse sequence of the form

$$C_{i_0} \xleftarrow{\lambda_{i_0+1}\circ\cdots\circ\lambda_{i_1}} C_{i_1} \xleftarrow{\lambda_{i_1+1}\circ\cdots\circ\lambda_{i_2}} C_{i_2} \xleftarrow{\lambda_{i_2+1}\circ\cdots\circ\lambda_{i_3}} \cdots .$$

In the future we will denote a composition $\lambda_i \circ \cdots \circ \lambda_j$ $(i \leq j)$ by $\lambda_{i,j}$.

Remark 1. Using the notation introduced in the previous subsection, a bonding map λ_i would be labeled λ_{i-1}^i and a composition $\lambda_i \circ \cdots \circ \lambda_j$ $(i \leq j)$ by λ_{i-1}^j. When working with inverse sequences, we opt for the slightly simpler notation described here.

Inverse sequences $\{C_i, \lambda_i\}$ and $\{D_i, \mu_i\}$ are isomorphic in pro-$\mathcal{C}$, or *pro-isomorphic*, if after passing to subsequences, there exists a commuting *ladder diagram*:

$$\begin{tikzcd}
C_{i_0} & & C_{i_1} \arrow[ll, "\lambda_{i_0+1,i_1}"'] \arrow[dl] & & C_{i_2} \arrow[ll, "\lambda_{i_1+1,i_2}"'] \arrow[dl] & & C_{i_3} \arrow[ll, "\lambda_{i_2+1,i_3}"'] \arrow[dl] & \cdots \\
& D_{j_0} \arrow[ul] & & D_{j_1} \arrow[ll, "\mu_{j_0+1,j_1}"'] \arrow[ul] & & D_{j_2} \arrow[ll, "\mu_{j_1+1,j_2}"'] \arrow[ul] & & \cdots \arrow[ll, "\mu_{j_2+1,j_3}"']
\end{tikzcd} \tag{2.1}$$

where the up and down arrows represent morphisms from $\mathcal{C}$. Clearly an inverse sequence is pro-isomorphic to any of its subsequences. To avoid tedious notation, we frequently do not distinguish $\{C_i, \lambda_i\}$ from its subsequences. Instead we simply assume that $\{C_i, \lambda_i\}$ has the desired properties of a preferred subsequence—often prefaced by the words "after passing to a subsequence and relabelling".

Remark 2. Together the collection of down arrows in (2.1) determine a morphism in pro-C from $\{C_i, \lambda_i\}$ to $\{D_i, \mu_i\}$ and the up arrows a morphism from $\{D_i, \mu_i\}$ to $\{C_i, \lambda_i\}$. Again see [Ge3, Ch.11] for details.

An inverse sequence $\{C_i, \lambda_i\}$ is *stable* if it is pro-isomorphic to a constant sequence

$$D \xleftarrow{\text{id}} D \xleftarrow{\text{id}} D \xleftarrow{\text{id}} \cdots .$$

For example, if each λ_i is an isomorphism from $\mathcal{C}$, it is easy to show that $\{C_i, \lambda_i\}$ is stable.

Inverse limits of an inverse sequences of sets are particularly easy to understand. In particular,

$$\varprojlim \{C_i, \lambda_i\} = \left\{ (c_0, c_1, c_2, \cdots) \in \prod_{i=0}^{\infty} C_i \,\middle|\, \lambda_i(c_i) = c_{i-1} \text{ for all } i \geq 1 \right\},$$

with a projection map $p_i : \varprojlim \{C_i, \lambda_i\} \to C_i$ for each $i \geq 0$.

2.3. **Inverse sequences of groups.** Of particular interest to us is the category $\mathcal{G}$ of groups and group homomorphisms. It is easy to show that an inverse sequence of groups $\{G_i, \lambda_i\}$ is stable if and only if, after passing to a subsequence and relabelling, there is a commutative diagram of the form

$$\begin{tikzcd}
G_0 & & G_1 \arrow[ll, "\lambda_1"'] \arrow[dl] & & G_2 \arrow[ll, "\lambda_2"'] \arrow[dl] & & G_3 \arrow[ll, "\lambda_3"'] \arrow[dl] & \cdots \\
& \mathrm{Im}(\lambda_1) \arrow[ul] & & \mathrm{Im}(\lambda_2) \arrow[ll, "\cong"'] \arrow[ul] & & \mathrm{Im}(\lambda_3) \arrow[ll, "\cong"'] \arrow[ul] & & \cdots \arrow[ll, "\cong"']
\end{tikzcd}$$

where all unlabeled maps are inclusions or restrictions. In this case $\varprojlim \{C_i, \lambda_i\} \cong \operatorname{im}(\lambda_i)$ and each projection homomorphism takes $\varprojlim \{C_i, \lambda_i\}$ isomorphically onto the corresponding $\operatorname{im}(\lambda_i)$.

The sequence $\{G_i, \lambda_i\}$ is *semistable* (or *Mittag-Leffler* or *pro-epimorphic*) if it is pro-isomorphic to an inverse sequence $\{H_i, \mu_i\}$ for which each μ_i is surjective. Equivalently, $\{G_i, \lambda_i\}$ is semistable if, after passing to a subsequence and relabelling, there

is a commutative diagram of the form

$$\begin{tikzcd}
G_0 & & G_1 \arrow[ll, "\lambda_1"'] \arrow[dl] & & G_2 \arrow[ll, "\lambda_2"'] \arrow[dl] & & G_3 \arrow[ll, "\lambda_3"'] \arrow[dl] & \cdots \\
& \mathrm{Im}\,(\lambda_1) \arrow[ul] & & \mathrm{Im}\,(\lambda_2) \arrow[ul] \arrow[ll, two heads] & & \mathrm{Im}\,(\lambda_3) \arrow[ul] \arrow[ll, two heads] & & \cdots \arrow[ll, two heads]
\end{tikzcd}$$

where "$\twoheadleftarrow$" denotes a surjection.

2.4. **Inverse systems and sequences of CW complexes.** Another category of utmost interest to us is $\mathcal{FH}_0$, the category of pointed, connected, finite CW complexes and pointed homotopy classes of maps. (A space is *pointed* if a basepoint has been chosen; a map is *pointed* if basepoint is taken to basepoint.) We will frequently refer to pointed spaces and maps without explicitly mentioning the basepoints. We will refer to an inverse system [resp., tower] from $\mathcal{FH}_0$ as an *inverse system* [resp., *tower*] *of finite complexes.*

For each $k \geq 1$, there is an obvious functor from pro-$\mathcal{FH}_0$ to pro-$\mathcal{G}$ taking an inverse system $\left\{K_\alpha, g_\alpha^\beta; \Omega\right\}$ of pointed, connected, finite simplicial complexes to be the inverse system of groups $\left\{\pi_k(K_\alpha), (g_\alpha^\beta)_*; \Omega\right\}$ (the k^{th} *homotopy pro-group of* $\left\{K_\alpha, g_\alpha^\beta; \Omega\right\}$). A related functor takes $\left\{K_\alpha, g_\alpha^\beta; \Omega\right\}$ to the group $\varprojlim \left\{\pi_k(K_\alpha), (g_\alpha^\beta)_*; \Omega\right\}$ which we denote $\check{\pi}_k \left(\left\{\pi_k(K_\alpha), (g_\alpha^\beta)_*; \Omega\right\}\right)$ (the k^{th} *Čech homotopy group of* $\left\{K_\alpha, g_\alpha^\beta; \Omega\right\}$).

Clearly the initial functor described above takes towers from pro-$\mathcal{FH}_0$ to towers in pro-$\mathcal{G}$, while the latter takes each tower to a group.

2.5. **Homotopy dimension.** The *dimension*, $\dim(\{K_i, f_i\})$, of a tower of finite complexes is the supremum (possibly ∞) of the dimensions of the K_i's. The *homotopy dimension* of $\{K_i, f_i\}$ is defined by:

$$\operatorname{hom}\dim\left(\{K_i, f_i\}\right) = \inf\{\dim(\{L_i, g_i\}) \mid \{L_i, g_i\} \text{ is pro-isomorphic to } \{K_i, f_i\}\}.$$

2.6. **Shapes of compacta.** Our view of *shape theory* is that it is the study of (possibly bad) compact metric spaces through the use of associated inverse systems and sequences of finite complexes.

Let Z be a compact, connected, metric space with basepoint z. Let Ω denote the set of all finite open covers $\mathcal{U}_\alpha$ of Z, each with a distinguished element U^* containing z. Declare $\mathcal{U}_\alpha \leq \mathcal{U}_\beta$ to mean that $\mathcal{U}_\beta$ refines $\mathcal{U}_\alpha$. Using Lebesgue numbers, it is easy to see that Ω is a directed set. For each $\mathcal{U}_\alpha$, let N_α be its nerve, and for each $\mathcal{U}_\alpha \leq \mathcal{U}_\beta$ let $g_\alpha^\beta : N_\beta \to N_\alpha$ be (the pointed homotopy class of) an induced simplicial map. In this way, we associate to Z an inverse system $\left\{N_\alpha, g_\alpha^\beta; \Omega\right\}$ from pro-$\mathcal{FH}_0$. We may then define pro-$\pi_k(Z)$ (the k^{th} *pro-homotopy group of* Z) to be the inverse system $\left\{\pi_1(N_\alpha), (g_\alpha^\beta)_*; \Omega\right\}$ and $\check{\pi}_k(Z)$ (the k^{th} *Čech homotopy group of* Z) its inverse limit.

Any cofinal tower contained in the above inverse system will be called a *tower of finite complexes associated to* Z. Another application of Lebesgue numbers shows that such towers always exist. We say that Z and Z' have the same *pointed shape* if their associated towers are pro-isomorphic. The *shape dimension* of Z is defined to

be the homotopy dimension of an associated tower. It is easy to see that the shape dimension of Z is less than or equal to its topological dimension.[1]

Since associated towers $\{N_i, g_i\}$ for Z are, by definition, cofinal subsystems of $\left\{N_\alpha, g_\alpha^\beta; \Omega\right\}$, each comes with a canonical isomorphism

$$j : \check{\pi}_k\left(\{N_i, g_i\}\right) = \varprojlim\left\{\pi_k(N_i), (g_i)_*\right\} \to \varprojlim\left\{\pi_k(N_\alpha), (g_\alpha^\beta)_*; \Omega\right\} \equiv \check{\pi}_k\left(Z\right).$$

2.7. **The reduced projective class group.** If Λ is a ring, we say that two finitely generated projective Λ-modules P and Q are *stably equivalent* if there exist finitely generated free Λ-modules F_1 and F_2 such that $P \oplus F_1 \cong Q \oplus F_2$. Under the operation of direct sum, the stable equivalence classes of finitely generated projective modules form a group $\widetilde{K}_0\left(\Lambda\right)$, known as the *reduced projective class group* of Λ. In this group, a finitely generated projective Λ-module P represents the trivial element if and only if it is *stably free*, i.e., there exists a finitely generated free Λ-module F such that $P \oplus F$ is free.

Of particular interest is the case where G is a group and Λ is the group ring $\mathbb{Z}\left[G\right]$. Then $\widetilde{K}_0$ determines a functor from the category $\mathcal{G}$ of groups to the category $\mathcal{AG}$ of abelian groups. In particular, a group homomorphism $\lambda : G \to H$ induces a ring homomorphism $\mathbb{Z}\left[G\right] \to \mathbb{Z}\left[H\right]$, which induces a group homomorphism $\lambda_* : \widetilde{K}_0\left(\mathbb{Z}\left[G\right]\right) \to \widetilde{K}_0\left(\mathbb{Z}\left[H\right]\right)$.

3. Main Results

We are now ready to state and prove the main results of this paper.

Theorem 3.1. *Let $\{K_i, f_i\}$ be a finite-dimensional tower of pointed, connected, finite complexes having stable* pro-π_k *for all k. Then there is a well defined obstruction $\omega\left(\{K_i, f_i\}\right) \in \widetilde{K}_0\left(\mathbb{Z}[\check{\pi}_1\left(\{K_i, f_i\}\right)]\right)$ which vanishes if and only if $\{K_i, f_i\}$ is stable in* pro-$\mathcal{FH}_0$.

Translating Theorem 3.1 into the language of shape theory yields the desired solution to Problem B:

Theorem 3.2. *A connected compactum Z with finite shape dimension has the pointed shape of a finite CW complex if and only if each of its homotopy pro-groups is stable and an intrinsically defined Wall obstruction $\omega\left(Z\right) \in \widetilde{K}_0\left(\mathbb{Z}[\check{\pi}_1\left(Z\right)]\right)$ vanishes.*

Our proof of Theorem 3.1 begins with two lemmas. The first is a simple and well known algebraic observation.

Lemma 3.3. *Let C_* be a chain complex of finitely generated free Λ-modules, and suppose that $H_i\left(C_*\right) = 0$ for $i \leq k$. Then*

(1) $\ker \partial_i$ *is finitely generated and stably free for all $i \leq k+1$, and*
(2) $H_{k+1}\left(C_*\right)$ *is finitely generated.*

[1] Another method for associating a tower of finite complexes to Z is to realize Z as the inverse limit of such complexes. It is a standard fact in shape theory that such a sequence will be pro-isomorphic to the ones obtained above. See, for example, [Bo] or [MS].

Proof. For the first assertion, begin by noting that $\ker \partial_0 = C_0$ is finitely generated and free. Proceeding inductively for $j \leq k+1$, assume that $\ker \partial_{j-1}$ is finitely generated and stably free. Since $H_{j-1}(C_*)$ is trivial, we have a short exact sequence

$$0 \to \ker \partial_j \to C_j \to \ker \partial_{j-1} \to 0.$$

By our assumption on $\ker \partial_{j-1}$, the sequence splits. Therefore, $\ker \partial_j \oplus \ker \partial_{j-1} \cong C_j$, which implies that $\ker \partial_j$ is finitely generated and stably free.

The second assertion follows from the first since $H_{k+1}(C_*) = \ker \partial_{k+1} \diagup \operatorname{im} \partial_{k+2}$. $\square$

The second lemma—which is really the starting point to our proof of Theorem 3.1—was extracted from [Fe, Th. 4]. It uses the following standard notation and terminology. For a map $f : K \to L$, the mapping cylinder of f will be denoted $M(f)$. The relative homotopy and homology groups of the pair $(M(f), K)$ will be abbreviated to $\pi_i(f)$ and $H_i(f)$. We say that f is *k-connected* if $\pi_i(f) = 0$ for all $i \leq k$; or equivalently, $f_* : \pi_i(K) \to \pi_i(L)$ is an isomorphism for $i < k$ and a surjection when $i = k$. The universal cover of a space K will be denoted $\widetilde{K}$. If $f : K \to L$ induces a π_1-isomorphism, then $\widetilde{f} : \widetilde{K} \to \widetilde{L}$ denotes a lift of f.

Lemma 3.4 (The Tower Improvement Lemma)**.** *Let $\{K_i, f_i\}$ be a tower of pointed, connected, finite complexes with stable* pro-π_k *for $k \leq n$ and semistable* pro-π_{n+1}*. Then there is a pro-isomorphic tower $\{L_i, g_i\}$ of finite complexes with the property that each g_i is $(n+1)$-connected. Moreover, after passing to a subsequence of $\{K_i, f_i\}$ and relabelling, we may assume that:*

1. *each L_i is constructed from K_i by inductively attaching finitely many k-cells for $2 \leq k \leq n+2$,*
2. *each g_i is an extension of f_i with $g_i(K_i \cup (\text{new cells of dimension } \leq k)) \subset (K_{i-1} \cup (\text{new cells of dimension } \leq k-1))$, and*
3. *the inclusions $K_i \hookrightarrow L_i$ form the promised pro-isomorphism from $\{K_i, f_i\}$ to $\{L_i, g_i\}$.*

Proof. Our proof is by induction on n.

Step 1. ($n = 0$) Let $\{K_i, f_i\}$ be a tower with semistable pro-π_1. By attaching 2-cells to the K_i's, we wish to obtain a new tower in which all bonding maps induce surjections on π_1.

By semistability, we may (by passing to a subsequence and relabelling) assume that each f_{i*} maps $f_{i+1*}(\pi_1(K_{i+1}))$ onto $f_{i*}(\pi_1(K_i))$. Let $\{{}^i a_j\}_{j=1}^{N_i}$ be a finite generating set for $\pi_1(K_i)$ and for each ${}^i a_j$ choose ${}^i b_j \in f_{i+1*}(\pi_1(K_{i+1}))$ such that $f_{i*}({}^i a_j) = f_{i*}({}^i b_j)$. For each element of the form ${}^i a_j ({}^i b_j)^{-1} \in \pi_1(K_i)$, attach a 2-cell to K_i which kills that element. Call the resulting complexes L_i's, and note that each f_i extends to a map $k_i : L_i \to K_{i-1}$. Define $g_i : L_i \to L_{i-1}$ to be k_i composed with the inclusion

$K_{i-1} \hookrightarrow L_{i-1}$. This leads to the following commutative diagram:

$$
\begin{tikzcd}
K_0 & & K_1 \arrow[ll, "f_1"'] \arrow[dl, hook] & & K_2 \arrow[ll, "f_2"'] \arrow[dl, hook] & & K_3 \arrow[ll, "f_3"'] \arrow[dl, hook] & \cdots \\
& L_1 \arrow[ul, "k_1"] & & L_2 \arrow[ll, "g_2"'] \arrow[ul, "k_2"] & & L_3 \arrow[ll, "g_3"'] \arrow[ul, "k_3"] & & \cdots \arrow[ll, "g_4"']
\end{tikzcd}
\tag{3.1}
$$

which ensures that the tower $\{L_i, g_i\}$ is pro-isomorphic to the original via inclusions.

Note that each $g_{i+1*} : \pi_1(L_{i+1}) \to \pi_1(L_i)$ is surjective. Indeed, the loops in K_i corresponding to the generating set $\{{}^i a_j\}$ of $\pi_1(K_i)$ still generate $\pi_1(L_i)$; moreover, in $\pi_1(L_i)$ each ${}^i a_j$ becomes identified with ${}^i b_j$ which lies in $\mathrm{im}\,(g_{i+1*})$. Properties 1 and 2 are immediate from the construction.

Step 2. $(n > 0)$ Now suppose $\{K_i, f_i\}$ is a tower such that pro-π_k is stable for all $k \leq n$ and pro-π_{n+1} is semistable.

We may assume inductively that there is a tower $\{L_i', g_i'\}$ which has n-connected bonding maps and (after passing to a subsequence of $\{K_i, f_i\}$ and relabelling) satisfies:

1′. each L_i' is constructed from K_i by inductively attaching finitely many k-cells for $2 \leq k \leq n+1$, and

2′. each g_i' is an extension of f_i such that $g_i'\,(K_i \cup (\text{new cells of dimension } \leq k)) \subset (K_{i-1} \cup (\text{new cells of dimension } \leq k-1))$.

3′. $\{L_i', g_i'\}$ and $\{K_i, f_i\}$ are pro-isomorphic via inclusions.

Since pro-π_{n+1} is semistable, we may also assume that:

4′. g_{i*}' maps $g_{i+1*}'(\pi_{n+1}(L_{i+1}'))$ onto $g_{i*}'(\pi_{n+1}(L_i'))$ for all i.

Since the g_i''s are n-connected, then each $g_{i*}' : \pi_k\,(L_i') \to \pi_k\left(L_{i-1}'\right)$ is an isomorphism for $k < n$. In addition, each $g_{i*}' : \pi_n\,(L_i') \to \pi_n\left(L_{i-1}'\right)$ is surjective; but since pro-π_n is stable, all but finitely many of these surjections must be isomorphisms. So, by dropping finitely many terms and relabelling, we assume that these also are isomorphisms.

Our goal is now clear—by attaching $(n+2)$-cells to the L_i''s, we wish to make each bonding map $(n+1)$-connected.

Due to the π_n-isomorphisms just established, we have an exact sequence

$$
\cdots \to \pi_{n+1}\,(L_i') \xrightarrow{g_{i*}'} \pi_{n+1}\left(L_{i-1}'\right) \to \pi_{n+1}\,(g_i') \to 0, \tag{3.2}
$$

for each i. Furthermore, since $n \geq 1$, each g_i' induces a π_1-isomorphism, so we may pass to the universal covers to obtain (by covering space theory and the Hurewicz theorem) isomorphisms:

$$
\pi_{n+1}\,(g_i') \cong \pi_{n+1}\,(\widetilde{g}_i') \cong H_{n+1}\,(\widetilde{g}_i')\,. \tag{3.3}
$$

Each term in the cellular chain complex $C_*\,(\widetilde{g}_i')$ is a finitely generated $\mathbb{Z}[\pi_1\,(L_i)]$-module; so, by Lemma 3.3, $H_{n+1}\,(\widetilde{g}_i')$ is finitely generated.

Applying (3.3), we may choose a finite generating set $\{{}^i\bar{\alpha}_j\}_{j=1}^{N_i}$ for each $\pi_{n+1}\,(g_i')$; and by (3.2), each ${}^i\bar{\alpha}_j$ may be represented by an ${}^i\alpha_j' \in \pi_{n+1}\left(L_{i-1}'\right)$. By Condition 3′ we may choose for each ${}^i\alpha_j'$, some ${}^i\beta_j \in \pi_{n+1}\,(L_i')$ such that $g_{i-1}' \circ g_i'\,({}^i\beta_j) = g_{i-1}'\left({}^i\alpha_j'\right)$.

Let ${}^i\alpha_j = {}^i\alpha'_j - g'_i\left({}^i\beta_j\right) \in \pi_{n+1}\left(L'_{i-1}\right)$. Then each ${}^i\alpha_j$ is sent to ${}^i\bar{\alpha}_j$ in $\pi_{n+1}\left(g'_i\right)$ and $g'_{i-1*}\left({}^i\alpha_j\right) = 0 \in \pi_{n+1}\left(L'_{i-2}\right)$. Attach $(n+2)$-cells to each L'_{i-1} to kill the ${}^i\alpha_j$'s. Call the resulting complexes L_i's, and for each i let $k_i : L_i \to L'_{i-1}$ be an extension of g'_i. Then let $g_i : L_i \to L_{i-1}$ be the composition of k_i with the inclusion $L'_{i-1} \hookrightarrow L_{i-1}$. This leads to a diagram like that produced in Step 1, hence the new system $\{L_i, g_i\}$ is pro-isomorphic to $\{L'_i, g'_i\}$, and thus to $\{K_i, f_i\}$ via inclusions. Moreover, it is easy to check that each g_i is $(n+1)$-connected. Properties 1 and 2 are immediate from the construction and the inductive hypothesis, and Property 3 from the final ladder diagram. □

Suppose now that $\{K_i, f_i\}$ has stable pro-π_k for all k. Then, by repeatedly attaching cells to the K_i's, one may obtain pro-isomorphic towers with r-connected bonding maps for arbitrarily large r. If $\{K_i, f_i\}$ is finite-dimensional it seems reasonable that, once r exceeds the dimension of $\{K_i, f_i\}$, this procedure will terminate with bonding maps that are connected in all dimensions—and thus, homotopy equivalences. Unfortunately, this strategy is too simplistic—in order to obtain r-connected maps we must attach $(r+1)$-cells; thus, the dimensions of the complexes continually exceeds the connectivity of the bonding maps. Roughly speaking, Theorem 3.1 captures the obstruction to making this strategy work.

Proof of Theorem 3.1. Begin with a tower $\{L_i, g_i\}$ of q-dimensional complexes pro-isomorphic to $\{K_i, f_i\}$, via a diagram of type (3.1), which has the following properties for all i.

a) g_i is $(q-1)$-connected,
b) for $k \in \{q-2, q-1, q\}$, g_i maps the k-skeleton of L_i into the $(k-1)$-skeleton of L_{i-1},
c) g_{i*} maps $g_{i+1*}(\pi_q(L_{i+1}))$ onto $g_{i*}(\pi_q(L_i))$, and
d) $g_{i*} : \pi_{q-1}(L_i) \to \pi_{q-1}(L_{i-1})$ is an isomorphism.

A tower satisfying Conditions a) and b) is easily obtainable; apply Lemma 3.4 to $\{K_i, f_i\}$ with $n = \dim\{K_i, f_i\} + 1$, in which case $q = \dim\{K_i, f_i\} + 3$. (**Note.** Although it may seem excessive to allow $\dim\{L_i, g_i\}$ exceed $\dim\{K_i, f_i\}$ by 3, this is done to obtain Condition b), which is key to our argument.) Semistability of pro-π_q gives Condition c)—after passing to a subsequence and relabeling. Then, since pro-π_{q-1} is stable and each $g_{i*} : \pi_{q-1}(L_i) \to \pi_{q-1}(L_{i-1})$ is surjective, we may drop finitely many terms to obtain Condition d).

As in the proof of Lemma 3.4, $\pi_q(g_i)$ and $H_q(\widetilde{g}_i)$ are isomorphic finitely generated $\mathbb{Z}[\pi_1 L_i]$-modules. We will show that, for all i, $H_q(\widetilde{g}_i)$ is projective and that all of these modules are stably equivalent. (This is a pleasant surprise, since the L_i's and g_i's may all be different.) Thus we obtain corresponding elements $[H_q(\widetilde{g}_i)]$ of $\widetilde{K}_0(\mathbb{Z}[\pi_1(L_i)])$. When these elements are trivial, i.e., when the modules are stably free, we will show that, by attaching finitely many $(q+1)$-cells to each L_i, bonding maps can be made homotopy equivalences. To complete the proof we define a single

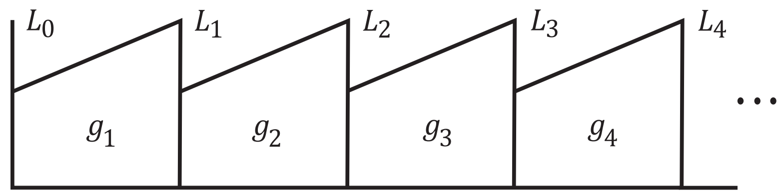

FIGURE 1.

obstruction $\omega(\{K_i, f_i\})$ to be the image of $(-1)^{q+1}[H_q(\widetilde{g}_i)]$ in $\widetilde{K}_0(\mathbb{Z}[\check{\pi}_1(\{K_i, f_i\})])$ and show that this element is uniquely determined by $\{K_i, f_i\}$.

Notes. **1)** To be more precise, the $H_q(\widetilde{g}_i)$ determine elements $(-1)^{q+1}[H_q(\widetilde{g}_i)]$ of $\widetilde{K}_0(\mathbb{Z}[\pi_1(L_i)])$ which may be associated, via inclusion maps (that induce π_1-isomorphisms), to elements of $\widetilde{K}_0(\mathbb{Z}[\pi_1(K_i)])$, which in turn determine a common element of $\widetilde{K}_0(\mathbb{Z}[\check{\pi}_1(\{K_i, f_i\})])$ via the projection maps—which in our setting are all isomorphisms.
2) We have used a factor $(-1)^{q+1}$ (instead of the more concise $(-1)^q$) so that our definition agrees with those already in the literature.

While most of our work takes place in the individual mapping cylinders $M(g_i)$ and their universal covers, there is some interplay between adjacent cylinders. For that reason, it is useful to view our work as taking place in the "infinite mapping telescope" shown in Figure 1 (and in its universal cover).

For ease of notation, fix i and consider the pair $\left(M(\widetilde{g}_i), \widetilde{L}_i\right)$. It is a standard fact (see [Co, 3.9]) that $C_*(\widetilde{g}_i)$ is isomorphic to the algebraic mapping cone of the chain homomorphism $g_{i*} : C_*\left(\widetilde{L}_i\right) \to C_*\left(\widetilde{L}_{i-1}\right)$. In particular, if the cellular chain complexes $C_*\left(\widetilde{L}_{i-1}\right)$ and $C_*\left(\widetilde{L}_i\right)$ of $\widetilde{L}_{i-1}$ and $\widetilde{L}_i$ are expressed as:

$$0 \to D_q \xrightarrow{d_q} D_{q-1} \xrightarrow{d_{q-1}} \cdots \xrightarrow{d_2} D_1 \xrightarrow{d_1} D_0 \longrightarrow 0, \quad \text{and} \tag{3.4}$$

$$0 \to D'_q \xrightarrow{d'_q} D'_{q-1} \xrightarrow{d'_{q-1}} \cdots \xrightarrow{d'_2} D'_1 \xrightarrow{d'_1} D'_0 \longrightarrow 0, \tag{3.5}$$

respectively, then $C_*(\widetilde{g}_i)$ is naturally isomorphic to a chain complex

$$0 \to C_{q+1} \xrightarrow{\partial_{q+1}} C_q \xrightarrow{\partial_q} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

where, for each j,

$$C_j = D'_{j-1} \oplus D_j \quad \text{and} \quad \partial_j(x, y) = (-\,d'_{j-1}x,\ \widetilde{g}_{i*}x + d_j y).$$

Here one views each $\pi_1(L_{i-1})$-module D_j as a $\pi_1(L_i)$-module in the obvious way—associating $a \cdot x$ with $\widetilde{g}_{i*}(a) \cdot x$ for $a \in \pi_1(L_i)$.

By Condition b), the map $\widetilde{g}_{i*} : D'_j \to D_j$ is trivial for $j \geq q-2$; so, in these dimensions, ∂_j splits as $-d'_{j-1} \oplus d_j$, allowing our chain complex to be written:

$$0 \to \underbrace{D'_q \oplus 0}_{C_{q+1}} \xrightarrow{-d'_q \oplus 0} \underbrace{D'_{q-1} \oplus D_q}_{C_q} \xrightarrow{-d'_{q-1} \oplus d_q} \underbrace{D'_{q-2} \oplus D_{q-1}}_{C_{q-1}} \xrightarrow{-d'_{q-2} \oplus d_{q-1}} \underbrace{D'_{q-3} \oplus D_{q-2}}_{C_{q-2}} \xrightarrow{\partial_{q-2}} \cdots .$$

Since the "minus signs" have no effect on kernels or images of maps, it follows that

$$\ker \partial_{q-1} = \ker(d'_{q-2}) \oplus \ker(d_{q-1}) \tag{3.6}$$

$$\ker \partial_q = \ker(d'_{q-1}) \oplus \ker(d_q) \tag{3.7}$$

$$\ker \partial_{q+1} = \ker(d'_q) \tag{3.8}$$

$$H_{q-1}(\widetilde{g}_i) = \left(\ker(d'_{q-2}) \diagup \operatorname{im}\left(d'_{q-1}\right)\right) \oplus \left(\ker(d_{q-1}) \diagup \operatorname{im}\left(d_q\right)\right) \tag{3.9}$$

$$H_q(\widetilde{g}_i) = \left(\ker(d'_{q-1}) \diagup \operatorname{im}\left(d'_q\right)\right) \oplus \ker(d_q) \tag{3.10}$$

$$H_{q+1}(\widetilde{g}_i) = \ker(d'_q) \tag{3.11}$$

Since $H_{q-1}(\widetilde{g}_i) = 0$, each summand in Identity 3.9 is trivial. Furthermore, the same reasoning applied to the adjacent mapping cylinder $M(g_{i+1})$ yields an analogous set of identities for $C_*(\widetilde{g}_{i+1})$ in which the "primed terms" become the "unprimed terms". This shows that $\ker(d'_{q-1}) \diagup \operatorname{im}\left(d'_q\right)$ is also trivial. Hence, the first summand in Identity 3.10 is trivial, so $H_q(\widetilde{g}_i) \cong \ker(d_q)$. Identity 3.7 together with Lemma 3.3 then shows that $H_q(\widetilde{g}_i)$ is finitely generated and projective. The same reasoning in $C_*(\widetilde{g}_{i+1})$ shows that $H_q(\widetilde{g}_{i+1}) \cong \ker(d'_q)$ is finitely generated projective, so by Identity 3.11, $H_{q+1}(\widetilde{g}_i)$ is finitely generated projective and naturally isomorphic to $H_q(\widetilde{g}_{i+1})$ (using $\widetilde{g}_{i+1*}$ to make $H_{q+1}(\widetilde{g}_i)$ a $\pi_1(L_{i+1})$-module).

Next we show that $H_q(\widetilde{g}_i)$ and $H_{q+1}(\widetilde{g}_i)$ are stably equivalent. Extract the short exact sequence

$$0 \to H_{q+1}(\widetilde{g}_i) \to D'_q \to \operatorname{im}\left(d'_q\right) \to 0$$

from above, then recall that $\operatorname{im}\left(d'_q\right)$ is equal to $\ker(d'_{q-1})$. The latter is projective, so

$$D'_q \cong H_{q+1}(\widetilde{g}_i) \oplus \ker(d'_{q-1}).$$

Thus $[H_{q+1}(\widetilde{g}_i)] = -\left[\ker(d'_{q-1})\right]$ in $\widetilde{K}_0(\pi_1(L_i))$. Since $[H_q(\widetilde{g}_i)] = [\ker(d_q)]$ and (by Identity 3.7), $[\ker(d_q)] = -\left[\ker(d'_{q-1})\right]$, $[H_q(\widetilde{g}_i)] = [H_{q+1}(\widetilde{g}_i)]$.

To summarize, we have shown that for each i:

- $H_q(\widetilde{g}_i)$ and $H_{q+1}(\widetilde{g}_i)$ are finitely generated and projective,
- $[H_q(\widetilde{g}_i)] = [H_{q+1}(\widetilde{g}_i)]$ in $\widetilde{K}_0(\pi_1(L_i))$, and
- $H_{q+1}(\widetilde{g}_i)$ naturally isomorphic to $H_q(\widetilde{g}_{i+1})$ as $\pi_1(L_{i+1})$-modules.

These observations combine to show that each $[H_q(\widetilde{g}_i)]$ determines the "same" element of $\widetilde{K}_0(\check{\pi}_1(\{K_i, f_i\}))$. More precisely, define $\omega(\{K_i, f_i\})$ to be the image of $(-1)^{q+1}[H_q(\widetilde{g}_i)]$ under the isomorphism $\widetilde{K}_0(\mathbb{Z}[\pi_1(L_i)]) \to \widetilde{K}_0(\check{\pi}_1(\{K_i, f_i\}))$ induced by the composition of group isomorphisms

$$\pi_1(L_i) \xrightarrow{p_i^{-1}} \varprojlim \{\pi_i(L_i), (g_i)_*\} \to \varprojlim \{\pi_i(K_i), (f_i)_*\} = \check{\pi}_1(\{K_i, f_i\}) \tag{3.12}$$

where $p_i : \varprojlim \{\pi_i(L_i), (g_i)_*\} \to \pi_1(L_i)$ is projection, and the isomorphism between inverse limits is canonically induced by ladder diagram (3.1).

Claim 1. If $\omega(\{K_i, f_i\}) = 0$, then $\{K_i, f_i\}$ is stable.

We will show that, by adding finitely many$q$- and $(q+1)$-cells to each of the above L_i's, we may arrive at a pro-isomorphic tower in which all bonding maps are homotopy equivalences.

By assumption, each $\mathbb{Z}\pi_1$-module $H_q(\widetilde{g}_i)$ becomes free upon summation with a finitely generated free module. This may be accomplished geometrically by attaching finitely many q-cells to the corresponding L_{i-1}'s via trivial attaching maps at the basepoints. Each g_{i-1} may then be extended by mapping these q-cells to the basepoint of L_{i-2}. Since this procedure preserves all relevant properties of our tower, we will assume that, for each i, $H_q(\widetilde{g}_i)$ (and therefore $\pi_q(g_i)$) is a finitely generated free $\mathbb{Z}[\pi_1(L_i)]$-module.

Proceed as in Step 2 of the proof of Lemma 3.4 to obtain collections $\{{}^i\alpha_j\}_{j=1}^{N_i} \subset \pi_{n+1}(L_{i-1})$ that correspond to generating sets for the $\pi_q(g_i)$'s and which satisfy $g_{i-1*}({}^i\alpha_j) = 0 \in \pi_q(L_{i-2})$ for all i, j. In addition, we now require that $\{{}^i\alpha_j\}_{j=1}^{N_i}$ corresponds to a free basis for $\pi_q(g_i)$. For each ${}^i\alpha_j$ attach a single $(q+1)$-cell to L_{i-1} to kill that element. Extend each g_i to $g_i' : L_i' \to L_{i-1}'$ as before, thereby obtaining a tower $\{L_i', g_i'\}$ for which all bonding maps are q-connected. Since the $(q+1)$-cells are attached to L_{i-1} along a free basis, we do not create any new $(q+1)$-cycles for the pair $\left(M(\widetilde{g}_i'), \widetilde{L}_i'\right)$, so no new $(q+1)$-dimensional homology is introduced. Moreover, the $(q+1)$-cells attached to L_{i-1} result in $(q+2)$-cells in $M(\widetilde{g}_{i-1}')$ which are attached in precisely the correct manner to kill $H_{q+1}(\widetilde{g}_{i-1})$ without creating any $(q+2)$-dimensional homology—this is due to the natural isomorphism discovered earlier between $H_{q+1}(\widetilde{g}_{i-1})$ and $H_q(\widetilde{g}_i)$. Thus the g_i''s are all $(n+2)$-connected, and since the L_i''s are $(n+1)$-dimensional, this means that the g_i''s are homotopy equivalences. So $\{L_i', g_i'\}$ and hence $\{K_i, f_i\}$, are stable in pro-$\mathcal{FH}_0$.

Claim 2. The obstruction is well defined.

We must show that $\omega(\{K_i, f_i\})$ does not depend on the tower $\{L_i, g_i\}$ and ladder diagram chosen at the beginning of the proof. First observe that any subsequence $\{L_{k_i}, g_{k_ik_{i-1}}\}$ of $\{L_i, g_i\}$ yields the same obstruction. This is immediate in the special case that $\{L_{k_i}, g_{k_ik_{i-1}}\}$ contains two consecutive terms of $\{L_i, g_i\}$. If not, notice that $\{L_{k_i}, g_{k_ik_{i-1}}\}$ is a subsequence of $L_{k_1} \leftarrow L_{k_1+1} \leftarrow L_{k_2} \leftarrow L_{k_3} \leftarrow \cdots$, which is a subsequence of $\{L_i, g_i\}$. Therefore the more general observation follows from the special case.

Next suppose that $\{L_i, g_i\}$ and $\{M_i, h_i\}$ are each towers of finite q-dimensional complexes satisfying the conditions laid out at the beginning of the proof. Then $\{L_i, g_i\}$ and $\{M_i, h_i\}$ are pro-isomorphic; so, after passing to subsequences and relabeling,

there exists a homotopy commuting diagram of the form:

$$\begin{tikzcd}
L_0 & & L_1 \arrow[ll, "g_1"'] \arrow[dl, "\mu_1"] & & L_2 \arrow[ll, "g_2"'] \arrow[dl, "\mu_2"] & & L_3 \arrow[ll, "g_3"'] \arrow[dl, "\mu_3"] & \cdots \\
& M_1 \arrow[ul, "\lambda_1"] & & M_2 \arrow[ll, "h_2"'] \arrow[ul, "\lambda_2"] & & M_3 \arrow[ll, "h_3"'] \arrow[ul, "\lambda_3"] & & \cdots \arrow[ll, "h_4"']
\end{tikzcd}$$

where all λ_i and μ_i are cellular maps. From here we may create a new tower:

$$M_1 \longleftarrow L_2 \longleftarrow M_4 \longleftarrow L_5 \longleftarrow M_7 \longleftarrow L_8 \longleftarrow M_{10} \longleftarrow \cdots$$

where the bonding maps are determined (up to homotopy) by the ladder diagram. Properties a),c) and d) hold for this tower due to the corresponding properties for $\{L_i, g_i\}$ and $\{M_i, h_i\}$. To see that Property b) holds, note that each bonding map is the composition of a g_i or an h_i with a cellular map. (This is why so many terms were omitted.) Since this new tower contains subsequences which are—up to homotopies of the bonding maps—subsequences of $\{L_i, g_i\}$ and $\{M_i, h_i\}$, our initial observation implies that all determine the same obstruction.

Finally we consider the general situation where $\{L_i, g_i\}$ and $\{M_i, h_i\}$ satisfy Conditions a)-d), but are not necessarily of the same dimension. By the previous case and induction, it will be enough to show that, for a given q-dimensional $\{L_i, g_i\}$, we can find a $(q+1)$-dimensional tower $\{L_i', g_i'\}$ which satisfies the corresponding versions of Conditions a)-d), and which determines the same obstruction as $\{L_i, g_i\}$. In this step, the need for the $(-1)^{q+1}$ factor finally becomes clear.

The tower $\{L_i', g_i'\}$ is obtained by carrying out our usual strategy of attaching a finite collection of $(q+1)$-cells to each L_{i-1} along a generating set for $H_q\left(M\left(\widetilde{g}_i\right), \widetilde{L}_i\right)$. The resulting $C_*\left(\widetilde{L}_i'\right)$'s differ from the $C_*\left(\widetilde{L}_i\right)$'s only in dimension $q+1$ where we have introduced finitely generated free modules ${}^iF_{q+1}$. By inserting this term into (3.4) and rewriting D_q as $\operatorname{im}(d_q) \oplus \ker(d_q)$, the chain complex for L_{i-1}' may be written:

$$0 \longrightarrow {}^iF_{q+1} \xrightarrow{d_{q+1}} \operatorname{im}(d_q) \oplus \ker(d_q) \xrightarrow{d_q} D_{q-1} \xrightarrow{d_{q-1}} \cdots \xrightarrow{d_2} D_1 \xrightarrow{d_1} D_0 \longrightarrow 0,$$

By construction, d_{q+1} takes ${}^iF_{q+1}$ onto $\ker(d_q)$ thereby eliminating the q-dimensional homology of the pair $\left(M\left(\widetilde{g}_i'\right), \widetilde{L}_i'\right)$. Note however, that we may have introduced new $(q+1)$-dimensional homology. Indeed, by our earlier analysis, $H_{q+1}(\widetilde{g}_i') = \ker(d_{q+1})$. (The original $(q+1)$-dimensional homology of the pair was eliminated—as it was in the unobstructed case—when we attached $(q+1)$-cells to L_i.) By extracting the short exact sequence

$$0 \longrightarrow \ker(d_{q+1}) \longrightarrow {}^iF_{q+1} \longrightarrow \ker(d_q) \longrightarrow 0$$

and recalling that $\ker(d_q) \cong H_q(\widetilde{g}_i)$ is projective, we have

$${}^iF_{q+1} \cong H_{q+1}(\widetilde{g}_i') \oplus H_q(\widetilde{g}_i).$$

So, upon projection into $\widetilde{K}_0(\check{\pi}_1(\{K_i, f_i\}))$, (as described in line (3.12)), $[H_{q+1}(\widetilde{g}_i')]$ and $-[H_q(\widetilde{g}_i)]$ determine the same element. The same is then true for $(-1)^{q+2}[H_{q+1}(\widetilde{g}_i')]$

and $(-1)^{q+1}[H_q(\widetilde{g}_i)]$, showing that $\{L_i, g_i\}$ and $\{L'_i, g'_i\}$ lead to the same obstruction. $\square$

Proof of Theorem 3.2. We need only verify the forward implication, as the converse is obvious.

Using the finite-dimensionality of Z, choose a finite-dimensional tower of pointed, connected, finite complexes $\{N_i, g_i\}$ associated to Z. By the pro-π_k hypotheses on Z, we may apply Theorem 3.1 to obtain $\omega(\{N_i, g_i\}) \in \widetilde{K}_0(\mathbb{Z}[\check{\pi}_1(\{N_i, g_i\})])$. The inclusion of $\{N_i, g_i\}$ into the associated inverse system $\{N_\alpha, g_\alpha^\beta; \Omega\}$, as described in §2.6, yields a canonical isomorphism of $\check{\pi}_1(\{K_i, g_i\}))$ onto $\check{\pi}_1(\{N_\alpha, g_\alpha^\beta; \Omega\}) = \check{\pi}_1(Z)$ which converts $\omega(\{N_i, g_i\})$ to our intrinsically defined Wall obstruction $\omega(Z) \in \widetilde{K}_0(\mathbb{Z}[\check{\pi}_1(Z)])$. $\square$

4. Realizing the obstructions

In addition to proving Theorems 3.1 and 3.2, Edwards and Geoghegan showed how to build towers and compacta with non-trivial obstructions. By applying their strategy within our framework, we obtain an easy proof of the following:

Proposition 4.1. *Let G be a finitely presentable group and P a finitely generated projective $\mathbb{Z}[G]$ module. Then there exists a tower of finite 2-complexes $\{K_i, f_i\}$, with stable* pro-π_k *for all k and $\check{\pi}_1(\{K_i, f_i\}) \cong G$, such that $\omega(\{K_i, f_i\}) = [P] \in \widetilde{K}_0(\mathbb{Z}[G])$.*

By letting $Z = \varprojlim \{K_i, f_i\}$ we immediately obtain:

Proposition 4.2. *Let G be a finitely presentable group and P a finitely generated projective $\mathbb{Z}[G]$ module. Then there exists a compact connected 2-dimensional pointed compactum Z, with stable* pro-π_k *for all k and $\check{\pi}_1(Z) \cong G$, such that $\omega(Z) = [P] \in \widetilde{K}_0(\mathbb{Z}[G])$.*

Proof. Let Q be a finitely generated projective $\mathbb{Z}[G]$ module representing $-[P]$ in $\widetilde{K}_0(\mathbb{Z}[G])$, and so that $F = P \oplus Q$ is finitely generated and free. Let r denote the rank of F. Let K' be a finite pointed 2-complex with $\pi_1(K') \cong G$, then construct K from K' by wedging a bouquet of r 2-spheres to K' at the basepoint. Then $\pi_2(K) \cong H_2\left(\widetilde{K}\right)$ has a summand isomorphic to F which corresponds to the bouquet of 2-spheres. Define a map $f : K \to K$ so that $f|_{K'} = \mathrm{id}$ and $f_* : \pi_2(K) \to \pi_2(K)$ (or equivalently $\widetilde{f}_* : H_2\left(\widetilde{K}\right) \to H_2\left(\widetilde{K}\right)$) is the projection $P \oplus Q \to P$ when restricted to the F-factor. Note that $H_2\left(\widetilde{f}\right) \cong Q \cong H_3\left(\widetilde{f}\right)$. Obtain the tower $\{K_i, f_i\}$ by letting $K_i = K$ for all $k \geq 0$ and $f_i = f$ for all $k \geq 1$.

To calculate $\omega(\{K_i, f_i\})$ according to the proof of Theorem 3.1, we must attach cells of dimensions $3, 4$ and 5 to each K_i to obtain an equivalent tower $\{L_i, g_i\}$ satisfying Conditions a)-d) of the proof. As we saw in Claim 2 of Theorem 3.1, this procedure simply shifts homology to higher dimensions. In particular, $[H_5(\widetilde{g}_i)] = -[H_2\left(\widetilde{f}\right)] = [P]$, as desired. $\square$

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